GROWTH OF CROSS-CHARACTERISTIC REPRESENTATIONS OF FINITE QUASISIMPLE GROUPS OF LIE TYPE

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ABSTRACT. In this paper we give a bound to the number of conjugacy classes of maximal subgroups of any almost simple group whose socle is a classical group of Lie type. The bound is $2n^{5.2} + n \log_2 \log_2 q$, where n is the dimension of the classical socle and q is the size of the defining field. To obtain the bound, we first bound the number of projective cross-characteristic representations of simple groups of Lie type as a function of the representation degree. These bounds are computed for different families of groups separately. In the computation, we use information on class numbers, minimal character degrees and gaps between character degrees.

1. INTRODUCTION

Let H be a finite quasisimple group, with $H/Z(H)$ a simple group of Lie type defined over a field of characteristic p. We are interested in the number of inequivalent n-dimensional irreducible modular representations of H. In [1], R. Guralnick, M. Larsen and P. H. Tiep obtain an upper bound of $n^{3.8}$ for the number of irreducible representations with dimension at most n in the defining characteristic p. They use their result to find an asymptotic bound for the number $m(G)$ of conjugacy classes of maximal subgroups of an almost simple group G with socle a group of Lie type. This bound is given as

$$m(G) < ar^6 + br \log \log q,$$

where a and b are unknown constants, and r and q are the rank of the socle and the size of its defining field, respectively.

In this paper, we sharpen the mentioned result in the case where the socle is a classical group, as follows:

Theorem 1.1. *Assume G is a finite almost simple group with socle a classical group of dimension n over the field $\mathbb{F}_q$. Let $m(G)$ denote the number of conjugacy classes of maximal subgroups of G not containing the socle. Then*

$$m(G) < 2n^{5.2} + n \log_2 \log_2 q.$$

Note that there are no unknown constants left in Theorem 1.1. To prove the result, we need to study the representation growth of groups of Lie type over fields of characteristic different from p. Define $r_n(H, \ell)$ as the number of inequivalent irreducible non-trivial n-dimensional representations of H over the algebraic closure of the finite prime field $\mathbb{F}_\ell$, with $\ell \neq p$. Also, write $r_n^f(H, \ell)$ for the number of such representations that are in addition faithful.

If $\mathcal{L}$ is a class of finite quasisimple groups of Lie type, we denote

$$s_n(\mathcal{L},\ell) = \sum_{H\in\mathcal{L}} r_n^f(H,\ell).$$

We will present upper bounds for the growth of $s_n(\mathcal{L},\ell)$ for different classes of groups of Lie type. The upper bounds will have no dependence on ℓ.

Regarding classical groups, we concern ourselves with the following families of quasisimple groups:

A_1	linear groups in dimension 2 (but see below)
A'	linear groups in dimension greater 2
${}^2A$	unitary groups in dimension greater than 2
B	orthogonal groups in odd dimension > 5 over a field of odd size
C	symplectic groups in dimension greater than 2
D	orthogonal groups of plus type in even dimension > 6
${}^2D$	orthogonal groups of minus type in even dimension > 6.

From family A_1, we also exclude groups $\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong \mathrm{Alt}(5)$, $\mathrm{PSL}_2(9) \cong \mathrm{Alt}(6)$, and all their covering groups.

Theorem 1.2. *We have for all $n > 1$ and for all ℓ, that*

$$s_n(A_1,\ell) \leq n+3.$$

Remark. For small n, the maximal value of $s_n(A_1,\ell)/n$ over ℓ can be computed from the known decomposition tables. It is then possible to state the largest obtainable values of this maximum over all n. The three largest ones are 7/6, 13/12 and 16/15. The largest appears for $n = 6$ and $n = 12$, and is used in the next theorem.

Theorem 1.3. *Let $\mathcal{L}$ denote one of the above families of finite quasisimple classical groups. For all $n > 1$ and for all ℓ, we have*

$$s_n(\mathcal{L},\ell) \leq c_{\mathcal{L}}\, n,$$

where the constants $c_{\mathcal{L}}$ are shown in Table 1.

$\mathcal{L}$:	A_1	A'	${}^2A$	B	C	D	${}^2D$
$c_{\mathcal{L}}$:	7/6	1.5484	2.8783	0.9859	2.8750	1.5135	1.7969

Table 1. Bounding constants for classical groups

Bounds in the last theorem are probably far from optimal (except for A_1), as can be judged from the known maximal values of $s_n(\mathcal{L},\ell)/n$ for $n \leq 250$ (see Table 8 on page 9).

The next theorem deals with the exceptional types. Let $\mathcal{E}$ denote the class of finite quasisimple groups with simple quotient an exceptional group of Lie type, excluding the groups $G_2(2)'$ and ${}^2G_2(3)'$. (These are isomorphic to $\mathrm{SU}_3(3)$ and $\mathrm{SL}_2(8)$, respectively.)

Theorem 1.4. *For all $n > 1$ and for all ℓ, we have*

$$s_n(\mathcal{E},\ell) < 1.2795\, n.$$

It is straightforward to add together all constants pertaining to different families to get a bound for representation growth of groups of Lie type.

Corollary 1.5. *Let $\mathcal{Q}$ denote the class of all finite quasisimple groups of Lie type. For all $n > 1$ and for all ℓ, we have*

$$s_n(\mathcal{Q}, \ell) < 14.1\, n.$$

In Section 2 we discuss the results from the literature that we will need for proving the theorems related to representation growth (1.2, 1.3 and 1.4). Section 3 is devoted to proving Theorems 1.2 and 1.3, and Section 4 to proving Theorem 1.4. The Main Theorem 1.1 is proved in Section 5.

Acknowledgments. The author wishes to thank his supervisor Professor Martin Liebeck for tireless guidance and support, and the Imperial College London for providing an excellent working environment. The author is also grateful to have been supported financially by Jenny and Antti Wihuri Foundation and the Engineering and Physical Sciences Research Council (DTA grant EP/P504694/1).

2. Preliminaries

Let H be a finite quasisimple group of Lie type, defined over a field $\mathbb{F}_q$. Then H is a factor group of the full covering group of the simple group $H/Z(H)$ of the same Lie type. All linear representations of H are included in the representations of the full covering group, so we may restrict our attention to this group.

The full covering group is completely determined by its simple quotient, which in turn is determined by its Lie family, rank r and the field size q. We will denote the full covering group by $H_r(q)$, where H is replaced by the letter of the Lie family in question. For example, $A_2(3)$ is the group $\mathrm{SL}_3(3)$, and $E_6(4)$ is the triple cover of the finite simple group of Lie type E_6 over the field of four elements. Apart from finitely many exceptions, the full covering group of a simple group of Lie type is of simply-connected type. The exceptions (given e.g. in [2, Th. 5.1.4]) are $A_1(4)$, $A_1(9)$, $A_2(2)$, $A_2(4)$, $A_3(2)$, ${}^2A_3(2)$, ${}^2A_3(3)$, ${}^2A_5(2)$, $B_3(3)$, $C_2(2)$, $C_3(2)$, $D_4(2)$, ${}^2E_6(2)$, $F_4(2)$, $G_2(3)$, $G_2(4)$, ${}^2B_2(8)$ and ${}^2F_4(2)'$ (the Tits group).

The proofs rely mainly on two types of information: results on smallest possible representation degrees of the full covering groups, and bounds for their conjugacy class numbers. On the first topic, there is a good survey article by P. H. Tiep [3].

Landazuri and Seitz found lower bounds for the representation dimensions in [4], and the bounds were subsequently improved by Seitz and Zalesskii [5]. These bounds were given for each Lie type as functions of rank and the size of the defining field. The bounds have later been slightly improved by various authors.

It is generally the case that the group H has a few representations of the smallest degree n_1, maybe a few also of the degrees $n_1 + 1$ and $n_1 + 2$, after which there is a relatively large *gap* before the next degrees. Then, after a couple of degrees, there is again a gap before the next one, and so on. The size of the first gap is known for linear, unitary and symplectic

groups, and the most recent results can be found in [6], [7], [8] and [9]. We will constantly refer to degrees *below* and *above the gap* when we talk about these groups, but it will be made clear in the context which degrees belong to which class. For the other classical groups, we take the first gap to exist before the smallest dimension, so that the phrase "degrees below the gap" comes to mean the empty set in this case.

For example, any group $A_r(q) = \mathrm{SL}_{r+1}(q)$ with $r \geq 2$ has irreducible cross-characteristic representations of dimensions

$$\begin{aligned} \frac{q^{r+1}-q}{q-1} - \kappa_{r,q,\ell} &= q^r + q^{r-1} + \cdots + q - \kappa_{r,q,\ell} \\ \text{and} \quad \frac{q^{r+1}-1}{q-1} &= q^r + q^{r-1} + \cdots + q + 1, \end{aligned}$$

where $\kappa_{r,q,\ell}$ is either 0 or 1, depending on r, q and ℓ. The difference between these two dimensions is at most two, and they are both said to be "below the gap". The next dimension is "above the gap", given by a polynomial with degree at least $2r-2$.

Bounds for the numbers of conjugacy classes of classical groups were found by J. Fulman and R. Guralnick in [10]. These bounds are all of the form $k(H) \leq q^r + B_{\mathcal{L}} q^{r-1}$, where $B_{\mathcal{L}}$ is a constant depending on the classical family $\mathcal{L}$ to which H belongs. The values for $B_{\mathcal{L}}$ are given in Table 2. For the exceptional groups, as well as classical groups of rank less than 9, one can obtain the precise conjugacy class numbers from Frank Lübeck's online data ([11]), where he lists all complex character degrees and their multiplicities for many kinds of groups of Lie type with rank at most eight.

$\mathcal{L}$:	A	^2A	B	C	D	^2D
$B_{\mathcal{L}}$:	3	15	22	30	32	32

TABLE 2. Fulman–Guralnick bounds for class numbers of classical groups

Finally, G. Hiss and G. Malle have determined all cross-characteristic representations of quasisimple groups with degree at most 250 ([12, 13]). This enables us to deal with the small degrees separately and assume in the general case that the degree is greater than 250. The Atlas of Finite Groups [14] together with the Atlas of Brauer Characters [15] tell us which groups have all representation degrees below 250, so that these groups can be discarded from consideration. Also the groups with exceptional covering group can mostly be handled with the help of the Atlases.

For technical reasons, we need to handle the types A_1, $^2B_2$ and $^2G_2$ separately. The generic complex character tables for these groups are known, and the cross-characteristic decomposition matrices can be found in [16], [17], [18] and [19].

3. The classical groups

In this section, we prove Theorems 1.2 and 1.3. The latter theorem is proved first, and the groups of class A_1 are dealt with in the end of the section. We take ℓ, the characteristic of the representation space, to be fixed, and suppress the notation as $r_n(H,\ell) = r_n(H)$.

Assume that $H = H_r(q)$ is a full covering group of a classical simple group of rank $r > 1$, defined over $\mathbb{F}_q$, where the characteristic of $\mathbb{F}_q$ is not ℓ. Suppose also, unless otherwise mentioned, that H is not an exceptional cover, but of simply-connected type. In the general treatment, we also assume for convenience that H is not $A_2(3)$, $A_2(5)$, $A_3(3)$, $A_5(2)$, $A_5(3)$, ${}^2A_2(3)$, ${}^2A_2(4)$, ${}^2A_2(5)$, $C_2(3)$ or ${}^2D_4(2)$. These groups will be dealt with later using specific information. To get upper bounds for those ranks and field sizes that may yield representations of dimension n, we need to bound the possible representation degrees of $H_r(q)$ from below.

Firstly, there is a small number of small degrees of H below the gap. For fixed rank, these dimensions are given by some polynomials in q, whose degrees depend on the rank. The polynomials are listed in various sources and collected in Table 3, together with multiplicities.

group	dimension	multiplicity	reference
$A_r(q)$, $r > 1$	$\frac{q^{r+1}-q}{q-1} - \left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}$	1	[6]
	$\frac{q^{r+1}-1}{q-1}$	$(q-1)_{l'} - 1$	
${}^2A_r(q)$	$\frac{q^{r+1}-q(-1)^r}{q+1}$	1	[7]
	$\frac{q^{r+1}+(-1)^r}{q+1}$	$(q+1)_{l'} - 1$	
$C_r(q)$, q odd	$\frac{1}{2}(q^r-1)$	2	[8]
	$\frac{1}{2}(q^r+1)$	2	

Table 3. Smallest dimensions of representations of some classical groups. The symbol $\{0\ /\ 1\}$ means either 0 or 1, depending on r, q and ℓ.

For each classical type $\mathcal{L}$ in $\{A', {}^2A, C\}$, it is straightforward to find a lower bound for all the degree polynomials given in Table 3, and it is also easy to bound the multiplicity polynomials from above. The bounds that will be used are listed in Table 4, and respectively called $\varphi_{\mathcal{L},r}$ and $\psi_{\mathcal{L}}$. From the former, we can deduce an upper bound for those field sizes q, for which $\varphi_{\mathcal{L},r}(q) = n$ holds for a given n. These bounds are listed in the same table as $q^1_{\mathcal{L},r}(n)$. Similarly, upper bounds can be found for the rank, and these are listed as $r^1_{\mathcal{L}}(n)$.

The following two lemmata will be used in bounding the number of representations with dimension below the gap.

Lemma 3.1. *Let H be a full covering group of one of the types A_r, ${}^2A_r$, or C_r, with fixed Lie family and rank. Let f_1 and f_2 be two polynomials expressing dimensions of representations of $H = H(q)$ below the gap, as*

$\mathcal{L}$	$\varphi_{\mathcal{L},r}(q)$	$\psi_{\mathcal{L}}(q)$	$q^1_{\mathcal{L},r}(n)$	$r^1_{\mathcal{L}}(n)$
A'	q^r	q	$n^{1/r}$	$\log_2 n$
${}^2A$	$(q-1)^r$, $\frac{1}{2}q^r$	q	$n^{1/r}+1$, $(2n)^{1/r}$	$\log_2 n + 1$
C	$\frac{1}{3}q^r$	2		$\log_3 n + 1$

TABLE 4. Bounds for the smallest dimension polynomials and their multiplicities for some classical groups

given in Table 3. If $n = f_1(q_1) = f_2(q_2)$ is an integer, then $q_1 = q_2$ and $f_1 = f_2$.

Proof. With any type A_r, $r \geq 2$, we have for $i \in \{1, 2\}$ that

$$q^r < f_i(q) < (q+1)^r \quad \text{for all } q.$$

Hence, if $n = f_i(q)$, we must have $n^{1/r} - 1 < q < n^{1/r}$. There is, however, at most one integer q that fits. Also, the difference between any two dimension polynomials for A_r is uniformly either 1 or 2, so that $f_1(q) = f_2(q)$ only if $f_1 = f_2$.

The cases for ${}^2A_r$ and C_r are handled similarly. □

Lemma 3.2. *Let H be of a fixed family $\mathcal{L}$ listed in Table 4, with fixed rank r. Letting $r_n^<(q)$ denote the number of irreducible cross-characteristic representations of $H = H(q)$ with dimension n below the gap, we have*

$$\sum_q r_n^<(q) < \psi_{\mathcal{L}}(q^1_{\mathcal{L},r}(n)),$$

with $\psi_{\mathcal{L}}$ and $q^1_{\mathcal{L},r}(n)$ as in Table 4.

Proof. By the previous lemma, for each n there can be at most one value of q, such that n is a dimension-below-the-gap of an irreducible representation of $H(q)$. Also, there can be at most one dimension polynomial f such that $f(q) = n$, so there is at most one multiplicity corresponding to f. This multiplicity is in turn bounded from above by $\psi_{\mathcal{L}}(q)$. Furthermore, the polynomial $\psi_{\mathcal{L}}$ is non-decreasing, so an upper bound is obtained by considering the largest possible value of q. □

After the few smallest representation degrees there is the first gap, and the next degrees are significantly larger. Lower bound for these larger dimensions that lie above the gap, is here called the *gap bound*, and it is again a polynomial in q with degree depending on r. We list the relevant information on the gap bounds in Table 5. (The bounds might not hold for some groups excluded above.) Notice that for the groups not appearing in Table 3, we take the gap bound to be the Landazuri–Seitz–Zalesskii bound (or one of its refinements) for the smallest dimension of a non-trivial irreducible representation.

In Table 6, we list as $\gamma_{\mathcal{L},r}(q)$ some lower bounds for the gap bounds. These bounds in turn yield upper bounds for those values of q, for which

group	gap bound	remarks	ref.
$A'_r(q)$	$(q^r-1)\left(\frac{q^{r-1}-q}{q-1}-\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}\right)$	$r \geq 4$	[6]
	$(q-1)(q^2-1)/\gcd(3,q-1)$	$r=2$	
	$(q-1)(q^3-1)/\gcd(2,q-1)$	$r=3$	
${}^2A_r(q)$	$\frac{(q^{r+1}+1)(q^r-q^2)}{(q^2-1)(q+1)}-1$	r even	[8]
$r \geq 4$	$\frac{(q^{r+1}-1)(q^r-q)}{(q^2-1)(q+1)}$	r odd	
${}^2A_r(q)$	$\frac{1}{6}(q-1)(q^2+3q+2)$	$r=2,\ 3 \mid (q+1)$	[7]
$r<4$	$\frac{1}{3}(2q^3-q^2+2q-3)$	$r=2,\ 3 \nmid (q+1)$	
	$\frac{(q^2+1)(q^2-q+1)}{\gcd(2,q-1)}-1$	$r=3$	
$B_r(q)$	$\frac{(q^r-1)(q^r-q)}{q^2-1}$	$q=3,\ r \geq 4$	[20]
	$\frac{q^{2r}-1}{q^2-1}-2$	$q>3$	
$C_r(q)$	$\frac{(q^r-1)(q^r-q)}{2(q+1)}$		[8, 9]
$D_r(q)$	$\frac{(q^r-1)(q^{r-1}-1)}{q^2-1}$	$q \leq 3$	[20]
	$\frac{(q^r-1)(q^{r-1}+q)}{q^2-1}-2$	$q>3$	
${}^2D_r(q)$	$\frac{(q^r+1)(q^{r-1}-q)}{q^2-1}-1$	$r \geq 6$	[20]
	1026	$r=4,\ q=4$	
	151	$r=5,\ q=2$	
	2376	$r=5,\ q=3$	

Table 5. Gap bounds for the classical groups

$n = \gamma_{\mathcal{L},r}(q)$. These bounds are listed as $q^2_{\mathcal{L},r}(n)$. Similarly, we get upper bounds for the ranks and list those as $r^2_{\mathcal{L}}(n)$.

The following lemmata are used in bounding the number of representations with dimension above the gap. The first one is a simple estimate preventing us from summing over all integers when we cannot determine which of them are prime powers.

Lemma 3.3. *Let H be of a fixed family $\mathcal{L}$ with fixed rank. Let $r_n^>(q)$ denote the number of irreducible n-dimensional cross-characteristic representations of $H = H(q)$ for n above the gap. Also, assume that K is a strictly increasing polynomial such that $K(q)$ is an upper bound for the conjugacy class number of $H(q)$ for all q. Finally, let Q be any positive integer. Then we have the following estimate:*

$$\sum_{q \leq Q} r_n^>(q) \;<\; K(Q) + \frac{1}{2}\sum_{i=3}^{Q-1} K(i) + \sum_{i=1}^{[\log_2 Q]} K(2^i).$$

Here $[x]$ denotes the integral part of x.

$\mathcal{L}$	$\gamma_{\mathcal{L},r}(q)$	$q^2_{\mathcal{L},r}(n)$	$r^2_{\mathcal{L}}(n)$	remarks
A'	q^{2r-2}	$n^{1/(2r-2)}$	$\frac{1}{2}\log_2 n + 1$	$r \geq 4$
	$\frac{1}{4}q^3$	$(4n)^{1/3}$		$r = 2$, $q \geq 7$
	$\frac{1}{3}q^4$	$(3n)^{1/4}$		$r = 3$, $q \geq 4$
${}^2A$	$\frac{1}{2}q^{2r-2}$	$(2n)^{1/(2r-2)}$	$\frac{1}{2}\log_2 n + \frac{3}{2}$	$r \geq 4$
	$\frac{1}{6}q^3$	$(6n)^{1/3}$		$r = 2$
	$\frac{2}{5}\cdot q^4$	$(5n/2)^{1/4}$		$r = 3$, $q \geq 4$
B	q^{2r-2}	$n^{1/(2r-2)}$	$\frac{1}{2}\log_3 n + 1$	$r \geq 3$
C	$\frac{1}{4}q^{2r-1}$	$(4n)^{1/(2r-1)}$	$\frac{1}{2}\log_2 n + \frac{3}{2}$	
$D, {}^2D$	q^{2r-3}	$n^{1/(2r-3)}$	$\frac{1}{2}\log_2 n + \frac{3}{2}$	$r \geq 4$

TABLE 6. Approximations for the gap bounds of classical groups

Proof. Firstly, there are fewer n-dimensional irreducible representations of H than there are conjugacy classes. This gives $\sum_q r_n^>(q) < \sum_q K(q)$. Secondly, q can be either odd or a power of two. The binary powers are handled by the second sum on the right hand side.

For odd values of q, we have two possibilities. If Q is even, the integers from 3 to Q can be partitioned into pairs $(2l-1, 2l)$, and we have $K(2l) > K(2l-1)$ for all l, since K is strictly increasing. Thus,

$$\sum_{\substack{q \leq Q \\ q \text{ odd}}} K(q) \leq \sum_{l=2}^{Q/2} K(2l-1) = \frac{1}{2}\sum_{l=2}^{Q/2} 2K(2l-1) < \frac{1}{2}\sum_{i=3}^{Q} K(i).$$

On the other hand, if Q is odd, we have by the same argument

$$\sum_{\substack{q \leq Q \\ q \text{ odd}}} K(q) < K(Q) + \frac{1}{2}\sum_{i=3}^{Q-1} K(i).$$

In both cases, we see that the desired inequality holds. □

Lemma 3.4. *Keep the notation of the previous lemma. Additionally, denote $K(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_s x^s$. Writing $q_2(n)$ for the bound $q^2_{\mathcal{L},r}(n)$ appearing in Table 6, we have*

$$\sum_q r_n^>(q) < K(q_2(n)) + \frac{1}{2}\int_0^{q_2(n)} K(x)\,dx + \alpha_0 \log_2 q_2(n) + \sum_{j=1}^{s} \frac{\alpha_j \big(2q_2(n)\big)^j}{2^j - 1}.$$

Proof. The result follows directly from the lemma above. The first two terms on the right hand side are obvious. For the last two, consider

$$\sum_{i=1}^{M} K(2^i) = \sum_{i=1}^{M} \left(\alpha_0 + \sum_{j=1}^{s} \alpha_j 2^{ij} \right) = \alpha_0 M + \sum_{j=1}^{s} \alpha_j \sum_{i=1}^{M} (2^j)^i.$$

It only remains to set $M = [\log_2 q_2(n)]$ and apply the formula for a geometric sum. □

We now embark upon finding the growth bounds for classical groups.

Proof of Theorem 1.3 (excluding A_1). Fix a classical family $\mathcal{L}$, and let $H_r(q)$ denote the full covering group of type $\mathcal{L}$ with rank r, defined over $\mathbb{F}_q$. We need to find an upper bound for the ratio

$$Q_{\mathcal{L}}(n) = \frac{1}{n} \sum_r \sum_q r_n(H_r(q)).^1$$

For each classical type $\mathcal{L}$, some small ranks and field sizes may be disregarded because of isomorphisms between the small groups. Also, the groups in A_1 are handled in a separate proof after this one. In each case, the smallest applicable rank and field size are denoted r_0 and q_0, respectively, and they are recalled in Table 7.

$\mathcal{L}:$	A'	^2A	B	C	D	^2D
$r_0:$	2	2	3	2	4	4
$q_0:$	2	2	3	2	2	2

TABLE 7. Smallest applicable ranks and field sizes

Let us first compute the maxima of $Q_{\mathcal{L}}(n)$ for $n \leq 250$. Here we use the tables of Hiss and Malle from [13]. The results are shown in Table 8. As these values are all less than the corresponding bounds given in the statement of the Theorem, we may henceforth assume that $n > 250$.

A'	^2A	B	C	D	^2D
1/2	2/3	2/27	3/13	1/8	1/33

TABLE 8. Maximal values of $Q_{\mathcal{L}}(n)$ for $n \leq 250$

The following is a general discussion of the structure of the proof, which will later be carried out case-by-case for each family.

First we consider those groups that may yield a dimension n below the gap. This only applies to families A', ^2A and C. We let $r_n^<(H)$ denote the

[1] In this proof, all sums over r and q, unless otherwise mentioned, are implicitly taken to be over the positive integers and prime powers, respectively. However, those pairs (r, q) are skipped which correspond to groups that have been explicitly excluded from consideration.

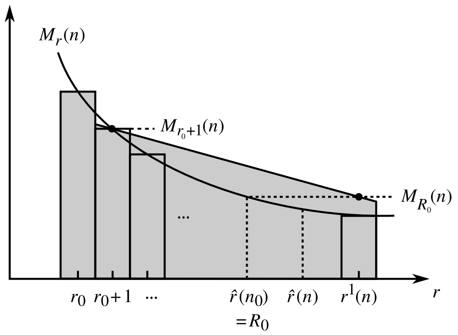

FIGURE 1. The trapezium used to obtain (2). Notice how the convexity of $M_r(n)$ guarantees that the shaded area is large enough to cover all the rectangles, even if $R_0 = r^1(n)$.

number of irreducible representations of H with dimension n below the gap, and desire a bound for the ratio

$$Q_{\mathcal{L}}^{<}(n) = \frac{1}{n} \sum_r \sum_q r_n^{<}(H_r(q)).$$

For a given rank r, let f_r denote the polynomial giving the smallest representation degree as exhibited in Table 3. Now, if $f_r(q_0) > n$, there is no group of type $\mathcal{L}$ and rank r that would have an irreducible representation of dimension n. On the other hand, if $f_r(q_0) \leq n$, the number of such representations is at most $\psi(q_r^1(n))$, according to Lemma 3.2. We obtain the following upper bound for $Q_{\mathcal{L}}^{<}(n)$:

$$R_{\mathcal{L}}^{<}(n) = \frac{1}{n} \sum_{r \geq r_0}{}' \psi(q_r^1(n)). \tag{1}$$

The prime ($'$) stands for taking the summand that corresponds to r into account only when $f_r(q_0) \leq n$ (this makes the sum finite).

The values of the expression $R_{\mathcal{L}}^{<}(n)$ can be computed explicitly for all values up to a desired n_0. After n_0, notice that ψ is linear and $q_r^1(n)$ has the form $(an)^{1/r}$. This means that the function $r \mapsto \psi(q_r^1(n))$ is decreasing and convex for all $n > n_0$, so we may use a trapezial estimate (see Figure 1) to bound the sum in (1) from above. The said sum is taken up to the biggest rank r for which $f_r(q_0) \leq n$. Denote this rank $\hat{r}(n)$. Since $f_r(q_0)$ is strictly increasing in r, also $\hat{r}(n)$ is increasing in n. So, for $n > n_0$, we are taking the sum at least up to $r = \hat{r}(n_0)$, denoted simply R_0. Now, write

$$M_r(n) = \psi(q_r^1(n)),$$

and let $r^1(n)$ denote the upper bound for the rank given in Table 4. Setting up a trapezium as in Figure 1, we get the following upper bound to $R_{\mathcal{L}}^{<}(n)$:

$$\bar{R}_{\mathcal{L}}^{<}(n) = \frac{1}{n} \left(M_{r_0}(n) + \frac{r^1(n) - r_0}{2} \big(M_{r_0+1}(n) + M_{R_0}(n) \big) \right). \tag{2}$$

Observe that $r^1(n)$ has the form $\log n + b$, whence it follows that $\bar{R}_{\mathcal{L}}^{<}(n)$ will be decreasing after some n. We will take care to choose n_0 above this point.

Let us then write $r_n^>(H_r(q))$ for the number of irreducible representations of $H_r(q)$ with dimension n greater than the gap bound. We are looking for a bound to

$$Q_{\mathcal{L}}^>(n) = \frac{1}{n} \sum_r \sum_q r_n^>(H_r(q)).$$

This bound will consist of two terms: $R_{\mathcal{L}}^1(n)$ and $R_{\mathcal{L}}^2(n)$, corresponding to small and large ranks, respectively.

For small ranks, that is, with r below some suitably chosen r_1, we can use precise conjugacy class numbers computed by Lübeck ([11]) to bound $r_n^>(H_r(q))$ from above. Write $k_r(q)$ for the class number of $H_r(q)$, and let $q_r^2(n)$ be the upper bound to q as given in Table 4. We define

$$R_{\mathcal{L}}^1(n) = \frac{1}{n} \sum_{r=r_0}^{r_1-1} {\sum_{q \geq q_0}}' k_r(q). \tag{3}$$

The latter sum is taken over those q, for which the gap bound of $H_r(q)$ is at most n, and there are only finitely many such q.

The values of $R_{\mathcal{L}}^1(n)$ can be computed for n up to a certain n_0. For larger values, we use a polynomial upper bound $K_r(q) = \sum_j \alpha_{r,j} q^j$ for the class number of $H_r(q)$. Although the class numbers are themselves given by polynomials in q, there can be finitely many different polynomials, each applicable to a certain congruence type of q. To obtain an upper bound, we simply choose the maximal one among these polynomials, and leave out any negative terms. It follows that the degree of K_r equals r, and all the coefficients $\alpha_{r,j}$ are non-negative integers.

The expression $R_{\mathcal{L}}^1(n)$ can now be bounded from above, in accordance with Lemma 3.4, by

$$\begin{aligned} \bar{R}_{\mathcal{L}}^1(n) = \frac{1}{n} \sum_{r=r_0}^{r_1-1} \Bigg(& K_r\big(q_r^2(n)\big) + \frac{1}{2} \int_0^{q_r^2(n)} K_r(x)\, dx \\ & + \alpha_0 \log_2 q_r^2(n) + \sum_{j \geq 1} \frac{\alpha_{r,j} \big(2 q_r^2(n)\big)^j}{2^j - 1} \Bigg). \end{aligned} \tag{4}$$

The important thing to note here is that for all types $\mathcal{L}$ and all ranks r, the largest power of n inside the outermost brackets is at most 1. (This can be seen by comparing the expression for $q_r^2(n)$ with the degree of K_r, which is r.) With all coefficients $\alpha_{r,j}$ non-negative, we can then be sure $\bar{R}_{\mathcal{L}}^1(n)$ is decreasing in n when $\log_2 q_r^2(n)/n$ is. Since $q_r^2(n)$ is of the form $(an)^{1/d_r}$ for some a and d_r, this happens after $n \geq e/a$.

For large ranks, that is, for $r \geq r_1$, we need to use the class number bounds of Fulman and Guralnick ([10]). With B given in Table 2, these bounds have the form $b_r(q) = q^r + Bq^{r-1}$. We write

$$R_{\mathcal{L}}^2(n) = \frac{1}{n} \sum_{r \geq r_1} {\sum_{q \geq q_0}}' \left(q^r + Bq^{r-1}\right).$$

The latter sum is again taken over those q, for which the gap bound of $H_r(q)$ is at most n.

Also the values of $R^2_{\mathcal{L}}(n)$ can be computed exactly up to a certain n_0. After that, we use the upper bound that results from substituting $b_r(q)$ in place of $K_r(q)$ in equation (4). When simplified, this becomes

$$\frac{1}{n}\sum_{r=r_1}^{[r^2(n)]}\Lambda_r(n),$$

where

$$\Lambda_r(n) = \frac{1}{2r+2}q_r^2(n)^{r+1} + \left(1+\frac{B}{2r}+\frac{2^r}{2^r-1}\right)q_r^2(n)^r + B\left(1+\frac{2^{r-1}}{2^{r-1}-1}\right)q_r^2(n)^{r-1}.$$

Notice that $q_r^2(n)$ has the form $(an)^{1/(br-c)}$ for some positive integers b and c. This makes every term in $\Lambda_r(n)$ decreasing in r (for any fixed n), except possibly the final $(an)^{\frac{r-1}{br-c}}$, which is increasing if and only if $b > c$. In this exceptional case (which occurs only when $\mathcal{L} = C$), we replace $(an)^{\frac{r-1}{br-c}}$ by its limit $(an)^{1/b}$. Then we can use the simple estimate

$$\bar{R}^2_{\mathcal{L}}(n) = \frac{r^2(n) - r_1 + 1}{n}\Lambda_{r_1}(n) \tag{5}$$

as an upper bound to $R^2_{\mathcal{L}}(n)$. Here, $r^2(n)$ is the upper bound for the rank as given in Table 6, and it has the form $a\log n + b$. Since the highest power of n in $\Lambda_{r_1}(n)$ is less than 1, this means that $\bar{R}^2_{\mathcal{L}}(n)$ will become decreasing after some n.

Finally, the ratio $Q_{\mathcal{L}}(n)$ is bounded above by the sum of $R^<_{\mathcal{L}}(n)$, $R^1_{\mathcal{L}}(n)$ and $R^2_{\mathcal{L}}(n)$. We shall finish the proof by computing the maximal values of these quantities for each classical family $\mathcal{L}$ separately. We also add the numbers arising from groups with exceptional covers individually in each case.

Case $\mathcal{L} = A'$. Let us first assume that (r, q) is none of $(2, 2)$, $(2, 3)$, $(2, 4)$, $(2, 5)$, $(3, 2)$, $(3, 3)$, $(5, 2)$ or $(5, 3)$. According to equation (1) and Table 4, we have

$$R^<_{A'}(n) = \sum_{r\geq r_0}{}' \frac{1}{n^{1-1/r}},$$

where the summand corresponding to r is taken into account if and only if $n \geq 2^{r+1} - 3$ (except for $r = 5$ where $q \geq 4$, so n has to be at least 1363). The values of $R^<_{A'}(n)$ will be computed explicitly up to $n_0 = 650000$. At n_0, the biggest r in the sum is 18, so looking at equation (2), we have

$$R^<_{A'}(n) < \bar{R}^<_{A'}(n) = \frac{1}{n^{1/2}} + \left(\frac{1}{2}\log_2 n - 1\right)\left(\frac{1}{n^{2/3}} + \frac{1}{n^{17/18}}\right)$$

for $n > n_0$. This expression is decreasing after n_0.

Let us then bound $Q^>_{A'}(n)$. We set $r_1 = 5$. For ranks less than r_1, we use Lübeck's class number polynomials to determine the values of $R^1_{A'}(n)$. The values of $R^1_{A'}(n)$ are computed up to n_0. After this, we substitute $q^2_{A',r}(n)$ from Table 6 into equation (4) to get a decreasing expression $\bar{R}^1_{A'}(n)$ that can be used as an upper bound.

We move on to large ranks. Write $g_r(q)$ for the gap bound of $A_r(q)$. For rank 5, we have assumed that $q \geq 4$, and $g_5(4) = 84909$. Thus, the smallest appearing gap bound is $g_6(2) = 1827$. For values of n from 1827 up to n_0, we will compute $R^2_{A'}(n)$, and afterwards we can substitute values from Tables 2 and 6 into equation (5) to obtain

$$\bar{R}^2_{A'}(n) = \left(\frac{1}{2}\log_2 n - 3\right)\left(\frac{1}{12n^{1/4}} + \frac{723}{310n^{3/8}} + \frac{31}{5n^{1/2}}\right).$$

This expression is decreasing for $n > n_0$.

Now, for n from 251 up to n_0, we can compute the exact values of

$$F(n) = R^<_{A'}(n) + R^1_{A'}(n) + R^2_{A'}(n),$$

and this bounds $Q_{A'}(n)$ from above. We are still, however, avoiding the exceptional cases. From the Atlases, we find that the full covering groups $A_2(2)$, $A_2(3)$, $A_2(4)$, $A_2(5)$ and $A_3(2)$ have no irreducible representations of dimension over 250, so we may forget about them. For $A_3(3)$, we extract the necessary data from the Atlases for directly computing $r_n(A_3(3))$ for all n. We find that the maximal value of $r_n(A_3(3))/n$ over all n is less than 0.01924.

For the remaining two groups, we use the following information obtainable from [6] and from the conjugacy class numbers of the groups, the latter of which are 60 for $A_5(2)$ and 396 for $A_5(3)$.

group	degree	multiplicity (at most)
$A_5(2)$	≥ 526	57
$A_5(3)$	362 or 363	1
	364	1
	≥ 6292	393

Having added to $F(n)$ the values of $r_n(A_3(3))/n$ (for all n), $57/n$ for $n \geq 526$, $1/n$ for $n \in \{362, 363, 364\}$, and $393/n$ for $n \geq 6292$, we find that the biggest total sum below n_0 appears at $n = 251$, and is less than 1.5484. More precisely, we have $R^<_{A'}(251) = 0.115$, $R^1_{A'}(251) = 1.435$, and $R^2_{A'}(251) = r_{251}(A_3(3)) = 0$.

For $n > n_0$, we use

$$\bar{F}(n) = \bar{R}^<_{A'}(n) + \bar{R}^1_{A'}(n) + \bar{R}^2_{A'}(n)$$

as a decreasing upper bound to $F(n)$. We have $\bar{F}(n_0) < 1.5267$, so even with the additions coming from the exceptional cases, we see that $F(n) < F(251)$ when $n > n_0$. Since according to Table 8, we have $Q_{A'}(n) < F(251)$ also for $n \leq 250$, we know that the obtained upper bound works for all n.

Case $\mathcal{L} = {}^2A$. We assume that (r, q) is none of $(2, 2)$ (soluble), $(2, 3)$, $(2, 4)$, $(2, 5)$, $(3, 2)$, $(3, 3)$ or $(5, 2)$. Proceeding in the same way as in the case $\mathcal{L} = A'$, we read from Table 4 that

$$R^<_{^2A}(n) = \sum_{r \geq r_0}{}' \left(\frac{1}{n^{1-1/r}} + \frac{1}{n}\right),$$

where each summand is taken into account when n is at least $(2^{r+1}-2)/3$, if r is even, or $(2^{r+1}-1)/3$, if r is odd. The values of $R^{<}_{2A}(n)$ will be computed explicitly up to $n_0 = 80000$. At n_0, the biggest rank is 16, so we use

$$\bar{R}^{<}_{2A}(n) = \frac{\sqrt{2}}{n^{1/2}} + (\log_2 n - 1)\left(\frac{1}{(2n)^{2/3}} + \frac{1}{(2n)^{15/16}}\right).$$

(Here we applied the second value given for $q^1_{2A,r}(n)$ in Table 4.) This function is decreasing for $n > n_0$.

To bound $Q^{>}_{2A}(n)$, we set $r_1 = 7$. For ranks less than r_1, we use Lübeck's class number polynomials to compute $R^1_{2A}(n)$ up to $n = n_0$. After this, we use $\bar{R}^1_{2A}(n)$, defined in (4), as a decreasing upper bound for $R^1_{2A}(n)$.

For $r \geq r_1$, the smallest gap bound is 3570 (the corresponding group is ${}^2A_7(2)$). For values of n from 3570 up to n_0, we will compute $R^2_{2A}(n)$, and afterwards we will use

$$\bar{R}^2_{2A}(n) = (\log_2 n - 9)\left(\frac{1}{16\cdot(2n)^{1/3}} + \frac{5475}{1778\cdot(2n)^{5/12}} + \frac{635}{21\cdot(2n)^{1/2}}\right).$$

This is decreasing in n for $n > n_0$.

For the exceptional cases, ${}^2A_2(3)$, ${}^2A_2(4)$, ${}^2A_2(5)$ and ${}^2A_3(2)$ have all degrees below 250, so this group can be ignored. The values of $r_n\big({}^2A_3(3)\big)/n$ can be computed using the Atlases, and the maximum is 0.03572. Finally, group ${}^2A_5(2)$ has 131 non-trivial conjugacy classes. Now, the values of

$$F(n) = R^{<}_{2A}(n) + R^1_{2A}(n) + R^2_{2A}(n) + \frac{r_n\big({}^2A_3(3)\big)}{n} + \frac{131}{n}$$

are computed up to n_0, and the maximal value is $F(272) < 2.8783$. (We have $R^{<}_{2A}(272) = 0.17$, $R^1_{2A}(272) = 2.24$ and $R^2_{2A}(272) = r_{272}({}^2A_3(3)) = 0$.) After n_0, the upper bound

$$\bar{F}(n) = \bar{R}^{<}_{2A}(n) + \bar{R}^1_{2A}(n) + \bar{R}^2_{2A}(n) + 0.03572 + 131/n$$

is decreasing, and $\bar{F}(n_0) < 2.873$.

Case $\mathcal{L} = B$. For the orthogonal groups, we have no gap results. Assume that (r,q) is not $(3,3)$. Also, we are assuming that $r \geq 3$ and $q \geq 3$, since otherwise we would be in coincidence with the symplectic groups. We set $r_1 = 5$. For ranks less than r_1, we compute $R^1_B(n)$ up to $n_0 = 58000$. When $n > n_0$, we use $\bar{R}^1_B(n)$ as a decreasing upper bound to $R^1_B(n)$.

For $r \geq r_1$, the smallest gap bound is 7260 (the corresponding group is $B_5(3)$). For values of n from 7260 up to n_0, we can compute $R^2_B(n)$, and afterwards we use the upper bound

$$\bar{R}^2_B(n) = \left(\frac{1}{2}\log_3 n - 3\right)\left(\frac{1}{12n^{1/4}} + \frac{656}{155n^{3/8}} + \frac{682}{15n^{1/2}}\right).$$

This is decreasing for $n > n_0$.

The exceptional case $B_3(3)$ has 87 non-trivial conjugacy classes. The values of

$$F(n) = R^1_B(n) + R^2_B(n) + \frac{87}{n}$$

are computed up to n_0, and the maximal value is $F(780) < 0.9859$. (We have $R_B^1(780) = 0.88$ and $R_B^2(780) = 0$.) After n_0, we apply

$$\bar{F}(n) = \bar{R}_B^1(n) + \bar{R}_B^2(n) + \frac{87}{n},$$

which is decreasing, and has $\bar{F}(n_0) < F(780)$.

Case $\mathcal{L} = C$. Assume (r, q) is none of $(2, 2)$, $(2, 3)$ or $(3, 2)$. Below the gap bound given in Table 5, representation dimensions exist only for groups with odd q. Referring to Table 4, we have

$$R_C^{<}(n) = a_n \cdot \frac{2}{n},$$

where a_n is the number of ranks $r \geq 2$ such that $n \geq (3^r - 1)/2$. The values of $R_C^{<}(n)$ will be computed up to $n_0 = 140000$. By using the bounds in Table 4 to approximate $a_n \leq \log_3(n)$, we obtain the upper bound

$$\bar{R}_C^{<}(n) = \frac{2 \log_3 n}{n}.$$

This is decreasing for $n > n_0$.

For $Q_C^{>}(n)$, we set $r_1 = 7$. For ranks less than r_1, we compute $R_C^1(n)$ up to n_0, and when $n > n_0$, we use $\bar{R}_C^1(n)$ as a decreasing upper bound to $R_C^1(n)$.

For $r \geq r_1$, the smallest gap bound is 2667 (the corresponding group is $C_7(2)$). For values of n from 2667 up to n_0, we can compute $R_C^2(n)$, and afterwards we use the upper bound

$$\bar{R}_C^2(n) = (\log_2 n - 9) \left(\frac{1}{8 \cdot (4n)^{5/13}} + \frac{7380}{889 \cdot (4n)^{6/13}} + \frac{5080}{21 n^{1/2}} \right).$$

(Note that the last term is chosen according to the discussion on page 12). This bound is decreasing for $n > n_0$.

For the exceptional cases, $C_2(2)$ and $C_2(3)$ have all degrees below 250. The values of $r_n(C_3(2))/n$ can be computed from the Atlases, and the maximum is 0.01072. The values of

$$F(n) = R_C^{<}(n) + R_C^1(n) + R_C^2(n) + \frac{r_n(C_3(3))}{n}$$

are computed up to n_0, and the maximal value is $F(288) = 2.8750$. (We have $R_C^{<}(288) = 1/36$, $R_C^1(288) = 205/72$ and $R_C^2(288) = r_{288}(C_3(3)) = 0$.) After n_0, the upper bound $\bar{F}(n) = \bar{R}_C^{<}(n) + \bar{R}_C^1(n) + \bar{R}_C^2(n)$ is decreasing, and $\bar{F}(n_0) < F(288)$.

Case $\mathcal{L} = D$. Assume (r, q) is not $(4, 2)$. There are no gap results. We set $r_1 = 8$. For ranks less than r_1, we compute $R_D^1(n)$ up to $n_0 = 222000$. When $n > n_0$, we use $\bar{R}_D^1(n)$ as a decreasing upper bound to $R_D^1(n)$.

For $r \geq r_1$, the smallest gap bound is 11048 (the corresponding group is $D_8(2)$). For values of n from 11048 up to n_0, we can compute $R_D^2(n)$, and afterwards we use the upper bound

$$\bar{R}_D^2(n) = (\log_2 n - 11) \left(\frac{1}{36 n^{4/13}} + \frac{1021}{510 n^{5/13}} + \frac{4080}{127 n^{6/13}} \right).$$

This is decreasing for $n > n_0$.

The exceptional group $D_4(2)$ is covered in the modular Atlas, so we can compute the values of $r_n(D_4(2))/n$, the maximum of which is 0.005792. The values of

$$F(n) = R^1_D(n) + R^2_D(n) + \frac{r_n(D_4(2))}{n}$$

are computed up to n_0, and the maximal value is $F(298) < 1.5135$. (We have $R^1_D(298) = F(298)$ and $R^2_D(298) = r_{298}(D_4(2)) = 0$.) After n_0, the upper bound $\bar{F}(n) = \bar{R}^1_D(n)+\bar{R}^2_D(n)+0.005792$ is decreasing, and $\bar{F}(n_0) < F(298)$.

Case $\mathcal{L} = {}^2D$. Assume (r, q) is not $(4, 2)$. There are no gap results. We set $r_1 = 7$. For ranks less than r_1, we compute $R^1_{2D}(n)$ up to $n_0 = 220000$. When $n > n_0$, we use $\bar{R}^1_{2D}(n)$ as a decreasing upper bound to $R^1_{2D}(n)$.

For $r \geq r_1$, the smallest gap bound is 2663 (the corresponding group is ${}^2D_7(2)$). For values of n from 2663 up to n_0, we can compute $R^2_{2D}(n)$, and afterwards we use the upper bound

$$\bar{R}^2_{2D}(n) = (\log_2 n - 9)\left(\frac{1}{32n^{3/11}} + \frac{3817}{1778n^{4/11}} + \frac{2032}{63n^{5/11}}\right).$$

This is decreasing for $n > n_0$.

The character degrees of ${}^2D_4(2)$ are given in the Atlases. The maximum of $r_n({}^2D_4(2))/n$ is 0.004202. The values of

$$F(n) = R^1_{2D}(n) + R^2_{2D}(n) + \frac{r_n({}^2D_4(2))}{n}$$

are computed up to n_0, and the maximal value is $F(251) < 1.7969$. (We have $R^1_{2D}(251) = F(251)$ and $R^2_{2D}(251) = r_{251}({}^2D_4(2)) = 0$.) After n_0, the upper bound $\bar{F}(n) = \bar{R}^1_{2D}(n) + \bar{R}^2_{2D}(n) + 0.004202$ is decreasing, and $\bar{F}(n_0) < F(251)$. $\square$

It only remains to check the case of rank one linear groups.

Proof of Theorem 1.2. We do not take groups $A_1(4) \cong A_1(5)$ or $A_1(9)$ into account, as they are isomorphic to alternating groups.

For $n \leq 250$, the given bound holds according to the tables of Hiss and Malle ([12]). Moreover, it can be computed that the largest value of $s_n(A_1)/n$ for $n \leq 250$ is $7/6$, and this can be obtained only at $n = 6$ and $n = 12$. Similarly, the second largest value $13/12$ can be obtained only at $n = 24$, and the third largest $16/15$ only at $n = 30$.

We may now assume that $q > 250$, since the maximal dimension of an irreducible representation of $A_1(q)$ is $q + 1$. With this assumption, the full cover of $A_1(q)$ is $\mathrm{SL}_2(q)$. For this group, the complex character degrees and their multiplicities are listed in Table 9.

The cross-characteristic decomposition numbers for the simple groups of type $\mathrm{PSL}_2(q)$ are given in [16]. If q is even, $\mathrm{SL}_2(q)$ is isomorphic to $\mathrm{PSL}_2(q)$, and if ℓ is even, $-1 \in \mathrm{SL}_2(q)$ acts trivially, so $\mathrm{SL}_2(q)$ has no faithful representations. Assume then that q and ℓ be odd. When q is not a power of ℓ, the Sylow ℓ-subgroups of $\mathrm{SL}_2(q)$ are cyclic. Now Dade's theorems can be used (see e.g. [21, §68]), and in this case they tell us that the decomposition numbers are all either 0 or 1, there are at most two irreducible

q even		q odd	
degree	multiplicity	degree	multiplicity
$q-1$	$q/2$	$q-1$	$(q-1)/2$
q	1	q	3
$q+1$	$(q-2)/2$	$q+1$	$(q-3)/2$
		$(q-1)/2$	2
		$(q+1)/2$	2

TABLE 9. Non-trivial complex character degrees of $\mathrm{SL}_2(q)$

Brauer characters in each block, and the set of irreducible ℓ-Brauer characters is a subset of the irreducible complex characters (restricted to p-regular elements). Therefore, it is enough to consider the case with $\ell = 0$.

We can read off from the character table that for any degree n, there are at most $n+3$ characters of this degree, of any groups of type SL_2. (Equality is obtained only if n, $n+1$, $n-1$, $2n+1$ and $2n-1$ are all prime powers.) Furthermore, when $n \geq 251$, we have $(n+3)/n < 1.012$. This proves Theorem 1.2, and also completes the proof of Theorem 1.3 above. □

4. Exceptional groups

In this section we shall prove Theorem 1.4. For full covering groups $H(q)$ of exceptional type, there are no gap results and the rank is always bounded. We will use Lübeck's conjugacy class numbers and the (sharpened) Landazuri–Seitz–Zalesskii bounds for the smallest representation degrees. The latter are listed in Table 10, where Φ_k denotes the k'th cyclotomic polynomial in q. As in the previous section, these lower bounds lead to upper bounds for the values of q for which $H(q)$ can have a representation of certain degree. Such upper bounds that will be used later are listed in Table 11 as $q^1_H(n)$.

Let $H(q)$ be a simply-connected full covering group of a simple group of an exceptional Lie type (the type is denoted by H alone), apart from the Suzuki and Ree types ${}^2B_2$ and ${}^2G_2$. Mimicking the previous section, we denote

$$Q_{\mathcal{E}}(n) = \frac{1}{n}\sum_H \sum_q r_n(H(q)),$$

where the first sum is over all exceptional types except the mentioned Suzuki and Ree types. Write $f_H(q)$ for the lower bound of the smallest representation dimension of $H(q)$ as given in Table 10. Now, $Q_{\mathcal{E}}(n)$ is bounded above by

$$R_{\mathcal{E}}(n) = \frac{1}{n}\sum_H {\sum_q}' k_H(q),$$

where $k_H(q)$ is the conjugacy class number of $H(q)$, and the dashed sum is taken over those q for which $H(q)$ is defined and $f_H(q) \leq n$.

group	bound for rep. degree	remark	ref.
$^2B_2(q)$	$(q-1)\sqrt{q/2}$		[4]
$^3D_4(q)$	$q^5 - q^3 + q - 1$		[22]
$E_6(q)$	$q(q^4+1)(q^6+q^3+1)-1$		[23]
$^2E_6(q)$	$(q^5+q)(q^6-q^3+1)-2$		[22]
$E_7(q)$	$q\Phi_7\Phi_{12}\Phi_{14}-2$		[23]
$E_8(q)$	$q\Phi_4^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}-3$		[23]
$F_4(q)$	$\frac{1}{2}q^7(q^3-1)(q-1)$	q even	[4]
	$q^6(q^2-1)$	q odd	[5]
$^2F_4(q)$	$q^4(q-1)\sqrt{q/2}$		[4]
$G_2(q)$	$q^2(q^2+1)$	$q \equiv 0 \pmod 3$	[3]
	q^3	$q \equiv 1 \pmod 3$	
	q^3-1	$q \equiv 2 \pmod 3$	
$^2G_2(q)$	$q(q-1)$		[4]

TABLE 10. Lower bounds for representation degrees of the full covering groups of exceptional groups of Lie type

H	$q_H^1(n)$	remark
$^3D_4$	$(4/3\cdot n)^{1/5}$	
E_6	$n^{1/11}$	
$^2E_6$	$(3/2\cdot n)^{1/11}$	$q \geq 3$
E_7	$n^{1/17}$	
E_8	$n^{1/29}$	
F_4	$(9/8\cdot n)^{1/8}$	
$^2F_4$	$(2n)^{2/11}$	$q \geq 8$
G_2	$(125/124\cdot n)^{1/3}$	$q \geq 5$

TABLE 11. Upper bounds for such q for which $H(q)$ may have a representation of degree n

For large n, we shall need the estimate $R_{\mathcal{E}}(n) < \bar{R}_{\mathcal{E}}(n)$, where $\bar{R}_{\mathcal{E}}(n)$ is defined analogously to $\bar{R}^1_{\mathcal{L}}(n)$ on page 11. The expression $\bar{R}_{\mathcal{E}}(n)$ becomes eventually decreasing. The main reason for this is that the ratio $\deg(k_H)/(\deg(f_H)+1)$ is at most 1, except for the two excluded types. (Note that for $H = {}^2F_4$, f_H is not actually a polynomial in q; in this case we say that $\deg(f_H) = 11/2$.)

We will now find the desired bound for the exceptional Lie types. Let us first deal with the excluded types.

Lemma 4.1. *For the Suzuki and Ree groups, if $n > 250$, we have*

$$\sum_q r_n\big({}^2B_2(q)\big) < \sqrt{n},$$

and

$$\sum_q r_n\big({}^2G_2(q)\big) < \sqrt[3]{n}.$$

Proof. Firstly, we can confirm from the Atlases that ${}^2B_2(8)$ has all irreducible representation degrees below 250. Recall that we exclude ${}^2G_2(2)$ and ${}^2G_2(3)$ because they are isomorphic to classical groups. (Notice also that ${}^2B_2(2)$ is soluble.) Thus we may assume that $q \geq 32$ for type ${}^2B_2$ and $q \geq 27$ for ${}^2G_2$. With these assumptions, the full covering groups are isomorphic to the simple groups.

Let us begin with the groups ${}^2B_2(q)$. The generic complex character table of ${}^2B_2(q)$ was introduced in [24], and the decomposition matrices for ℓ-modular representations of these groups are given in [17]. From this information, we see that the set of irreducible Brauer characters is a subset of the irreducible complex characters (restricted to p-regular elements), except when ℓ divides $q - \sqrt{2q} + 1$. In the latter case, there appears an additional irreducible Brauer character with degree $q^2 - 1$. The possible ℓ-modular degrees are all listed in Table 12. The multiplicities in the table for degrees apart from $q^2 - 1$ are for complex characters (this gives an upper bound for them).

degree	multiplicity	class
$(q-1)\sqrt{q/2}$	2	A
$(q-\sqrt{2q}+1)(q-1)$	$(q+\sqrt{2q})/4$	
q^2-1	1	
q^2	1	B
q^2+1	$q/2-1$	
$(q+\sqrt{2q}+1)(q-1)$	$(q-\sqrt{2q})/4$	

TABLE 12. Non-trivial character degrees of ${}^2B_2(q)$

The character degrees of ${}^2B_2(q)$ are given by polynomials in $\sqrt{q}$. These polynomials can be divided into two classes according to their degree, as shown in the last column of Table 12. The polynomials in class B are increasing and strictly between $\frac{3}{4}q^2$ and $2q^2$. Suppose now that f_1 and f_2 are two polynomials from class B, and that $f_1(q_1) = f_2(q_2)$ for some $q_1 < q_2$. As ${}^2B_2(q)$ is only defined for odd powers of 2, we know that $q_2 \geq 4q_1$. It follows that

$$f_1(q_1) < 2q_1^2 < \frac{3}{4}q_2^2 < f_2(q_2),$$

which is impossible. This means that if $f_1(q_1) = f_2(q_2)$ for two polynomials from class B, we must have $q_1 = q_2$. However, it is easily checked that for any q, different polynomials in class B give different values. Hence it follows

that if n is a character degree of ${}^2B_2(q)$, and $n \neq (q-1)\sqrt{q/2}$, then q is fixed and $n = f(q)$ for a unique polynomial in class B.

In the case just described (where n is a value of a polynomial from class B), we know that $n > \frac{3}{4}q^2$, so that $q < 2\sqrt{n}/\sqrt{3}$. Substituting this to the largest multiplicity polynomial in class B and adding the 2 possible characters coming from class A, we find that

$$\sum_{q \geq 32} r_n\big({}^2B_2(q)\big) < \frac{1}{2\sqrt{3}}\sqrt{n} + \frac{1}{2\sqrt[4]{3}}\sqrt[4]{n} + 2.$$

This gives the first result.

In case of ${}^2G$, a similar analysis can be made, based on the information on complex characters given in [25] and on decomposition matrices found in [18] and [19]. We omit the details. □

Proof of Theorem 1.4. Recall that we are not considering groups $G_2(2)$ or ${}^2G_2(3)$ here, as they are isomorphic to classical groups. From the tables of Hiss and Malle ([12]) we find that the stated bound holds for all $n \leq 250$.

Assume that $n > 251$. We first take care of the following exceptional cases: ${}^2E_6(2)$, $F_4(2)$, $G_2(3)$, $G_2(4)$, and ${}^2F_4(2)'$. The full covering groups ${}^2F_4(2)'$, $G_2(3)$ and $G_2(4)$ can be found in the Atlases. Letting $\mathcal{H}$ denote the set of these four groups, we can compute

$$\max_n \sum_{H \in \mathcal{H}} \frac{r_n(H)}{n} = 0.03389.$$

For those groups that are not covered in the modular Atlas, ${}^2E_6(2)$ has 346 conjugacy classes, and $F_4(2)$ has 95. The smallest non-trivial representation degree of ${}^2E_6(2)$ is at least 1536, according to [4].

We use Lemma 4.1 to estimate the contribution of types ${}^2B_2$ and ${}^2G_2$. Computing the values of

$$F(n) = R_{\mathcal{E}}(n) + \frac{1}{n}\left(\sum_{H \in \mathcal{H}} r_n(H) + [345] + 94 + \sqrt{n} + \sqrt[3]{n}\right)$$

up to $n_0 = 23000$, adding the [345] only while $n \geq 1536$, we find that the maximum is $F(251) < 1.27949$. (We have $R_{\mathcal{E}}(251) = 0.82$ and $r_{251}(H) = 0$ for all $H \in \mathcal{H}$.) Afterwards, the function

$$\bar{F}(n) = \bar{R}_{\mathcal{E}}(n) + 0.03389 + \frac{345 + 94}{n} + \frac{1}{\sqrt{n}} + \frac{1}{n^{2/3}}$$

is decreasing, $F(n) < \bar{F}(n)$, and $\bar{F}(n_0) < F(251)$. □

5. Main result

The Main Theorem 1.1 is proved along the same lines as Theorem 6.3 in [1]. Let G be an almost simple group having as its socle $G_0 = G_0(V, \mathbf{f})$, a classical simple group preserving the form $\mathbf{f}$ on the n-dimensional $\mathbb{F}_q$-space V. (The form is either trivial, or a non-degenerate alternating, quadratic or hermitian form, and the action is projective). With this notation, we have $G_0 \trianglelefteq G \leq \mathrm{Aut}(G_0)$, and apart from some known exceptions, the automorphism group is $\mathrm{P}\Gamma(V, \mathbf{f})$, the group of projective semilinear transformations preserving $\mathbf{f}$ up to scalar multiplication. The exceptions occur when G_0 is

one of $\mathrm{PSL}_n(q)$, $\mathrm{Sp}_4(2^k)$ or $\mathrm{P}\Omega_8^+(q)$, with additional (graph) automorphisms of order 2, 2 and 3, respectively (see [2, Th. 2.1.4]).

Let M be a maximal subgroup of G not containing G_0. It is known that the intersection $M \cap G_0$ is non-trivial (see e.g. [26, p. 395]), whence $M = N_G(M \cap G_0)$. It then follows that the intersection of M with the simple group determines M, and moreover, two maximal subgroups with non-trivial intersections are conjugate in G if and only if the intersections are. If all the subgroups have non-trivial intersections, then it suffices to count the G_0-classes of the intersections.

Assume now that $\mathrm{Aut}(G_0) = \mathrm{P}\Gamma(V, \mathbf{f})$ (the general case). To count the number of G_0-classes of maximal subgroups, we shall use Aschbacher's Theorem on maximal subgroups of classical groups ([27], see also [2]). By this theorem, M either belongs to one of so-called *natural families* $\mathcal{C}_1$–$\mathcal{C}_8$ of maximal subgroups, or in a further family consisting of almost simple groups that are absolutely irreducible on V. This family will be called $\mathcal{S}$.

It is often the number of Δ-classes of $G_0 \cap M$ which is easiest to bound, where Δ is the group of projective similarities of V preserving $\mathbf{f}$. In these cases, to account for the splitting of Δ-classes under G_0, we need to insert a factor of $[\Delta : G_0]$. In the linear and unitary groups, this index at most the dimension of V, otherwise it is at most 8.

For future calculations, we quote here a result of Lübeck.

Lemma 5.1 ([28], Th. 5.1)**.** *Let H be a full covering group of a classical group, and let ρ be a restricted, absolutely irreducible representation of H in the defining characteristic. If H is of type A_r or ${}^2A_r$ for some r, then either $\deg(\rho) > r^3/8$, or otherwise ρ is the natural representation or its dual, or one of at most 3 dual pairs of representations. If H is of another classical type of rank r, then either $\deg(\rho) > r^3$, or otherwise ρ is the natural representation or one of at most 2 representations.*

Before proceeding, we also mention the following information on the alternating groups.

Lemma 5.2. *Let G_d denote the alternating group $\mathrm{Alt}(d)$ or its full covering group. Then the number of d such that there exists an irreducible modular representation of G_d of dimension n, is less than $2\sqrt{n} + 3.81\log_2 n + 5$.*

Proof. For dimension $n \leq 250$, Hiss and Malle have listed all the representations of finite quasisimple groups in [12], so the claim can be readily checked in this case.

Assume that $n > 250$. When an irreducible representation of $\mathrm{Sym}(d)$ is restricted to $\mathrm{Alt}(d)$, it may stay irreducible or split into two irreducible representations of half the dimension. Define $f(d) = \frac{1}{2}(d-1)(d-2)$. By [29, Th. 7], all except three non-trivial irreducible representations of $\mathrm{Sym}(d)$ have dimension at least $f(d)$, provided that $d \geq 15$. The three exceptional representations do not split in $\mathrm{Alt}(d)$ when $d \geq 15$. Now, assuming that n is a dimension of an irreducible representation of $\mathrm{Alt}(d)$, at least one of the following must hold:

(1) $n \geq f(d)/2$
(2) $d \geq 15$ and n is one of three known values
(3) $d < 15$.

On the other hand, having $d < 15$ without $n \geq f(d)/2$ contradicts our assumption that $n > 250$. Hence either (1) or (2) holds, so the number of possible values of d, such that there exists an n-dimensional irreducible representation of $\mathrm{Alt}(d)$, is at most $f^{-1}(2n) + 3 < 2\sqrt{n} + 5$.

Let then $\hat{\mathrm{Alt}}(d)$ be the full covering group of $\mathrm{Alt}(d)$. We may assume that $d > 9$, since otherwise all representations have dimension less than 250. From [30], we find that the smallest dimension of a faithful irreducible representation of $\hat{\mathrm{Alt}}(d)$ is at least $g(d) = 2^{[(d-s-1)/2]}$, where s is the number of 1's in the binary expansion of d. Estimating $g(d) > 1.2^d$, we find that the number of d such that there exists a faithful n-dimensional irreducible representation of $\hat{\mathrm{Alt}}(d)$, is less than $3.81 \log_2 n$. The claim follows. $\square$

The following lemma deals with the exceptional automorphisms.

Lemma 5.3. *Retain the previous notation, with G, G_0 and Δ.*

(a) *Suppose $G_0 = \mathrm{Sp}_4(2^k)$ and $G \not\leq \mathrm{P\Gamma}(V, \mathbf{f})$. The number of G-conjugacy classes of maximal subgroups M of G, such that $G_0 \not\leq M$ and M does not appear in Aschbacher's classes $\mathcal{C}_5$ or $\mathcal{S}$, is at most* 5.

(b) *Suppose $G_0 = \mathrm{P}\Omega_8^+(q)$. The number of G-conjugacy classes of maximal subgroups M of G, such that $G_0 \not\leq M$ and M does not appear in class $\mathcal{C}_5$, is at most* 44.

Proof. (a) In [27, Section 14], a new system of families $\mathcal{C}_1'$–$\mathcal{C}_5'$ is given, which contains all maximal subgroups not included in $\mathcal{S}$. Moreover, $\mathcal{C}_4'$ is the same family as the old family $\mathcal{C}_5$. It is then proved that there is only one $\mathrm{Aut}(G_0)$-conjugacy class in each of $\mathcal{C}_1'$, $\mathcal{C}_3'$ and $\mathcal{C}_5'$. In $\mathcal{C}_2'$, there are at most two $\mathrm{Aut}(G_0)$-classes. In fact, the proof also shows that the classes do not split under G.

(b) This can be counted from [31, Section 1.5], where Kleidman exhibits all conjugacy classes of maximal subgroups of G. The subgroups of subfield type are denoted by S. $\square$

Now we are in a position to prove the main result of this paper. We refer to the notation presented in the beginning of this section.

Proof of Theorem 1.1. For the exceptional automorphisms, we assume that G_0 is neither one of the groups in parts (a) and (b) of Lemma 5.3. These cases are dealt with at the end of the proof. The only other exceptional automorphism groups are those of $G_0 = \mathrm{PSL}_n(q)$. Following Aschbacher's example, we simply replace the family $\mathcal{C}_1$ with a new one, $\mathcal{C}_1'$, as this is the only family that is affected.

Let us first look at Aschbacher's natural families.

Subfield family. Class $\mathcal{C}_5$ contains groups defined over a subfield of $\mathbb{F}_q$ of prime index. By [32, Lemma 2.1], the number of Δ-conjugacy classes of maximal subgroups (intersected with G_0) is at most $\log_2 \log_2 q + 1$. We need to add a factor of n to account for the splitting, so that the total number of the G_0-conjugacy classes of maximal subgroups of the subfield family becomes at most $n(\log_2 \log_2 q + 1)$.

Other natural families. We refer again to [32, Lemma 2.1]. There, an upper bound to the number of Δ-classes of maximal subgroups of non-subfield type is given as

$$\frac{3n}{2} + 4d(n) + \pi(n) + 3\log_2 n + 8,$$

where $d(n)$ is the number of divisors of n, and $\pi(n)$ the number of prime divisors. Multiplying by n and using the crude estimate $\pi(n) \leq d(n) \leq \log_2 n$, we find that the number of G_0-classes of maximal subgroups is at most $\frac{3}{2}n^2 + 8n\log_2 n + 8n$ in this case.

By Aschbacher's Theorem, any maximal subgroup not appearing in the natural families is itself an almost simple group whose socle is absolutely irreducible on V. Let M be one of these maximal subgroups and let M_0 denote its socle. Then we have $M_0 \leq M \cap G_0 \lhd M$ and $N_G(M_0) = M$. As before, it follows it is enough to consider the G_0-conjugacy classes of the socles.

The socles in turn correspond to absolutely irreducible modular projective representations of simple groups, and equivalent representations correspond to subgroups conjugate under $\mathrm{PGL}(V)$. Furthermore, the conjugating element will always reside in Δ (see [2, Cor. 2.10.4]). Imitating [1], we now divide the non-natural subgroups into five subfamilies:

- $\mathcal{S}_1$ the socle is an alternating group
- $\mathcal{S}_2$ the socle is a sporadic group
- $\mathcal{S}_3$ the socle is a group of Lie type in characteristic not dividing q
- $\mathcal{S}_4$ the socle is a group of Lie type in characteristic dividing q, and the representation is not restricted
- $\mathcal{S}_5$ the socle is a group of Lie type in characteristic dividing q, and the representation is restricted.

In each case, we will bound the number of inequivalent representations.

Case $\mathcal{S}_1$. By Lemma 5.2, the number of non-isomorphic $\mathrm{Alt}(d)$ that can yield an n-dimensional irreducible projective representation is at most $2\sqrt{n} + 3.81\log_2 n + 5$. On the other hand, it was shown in Theorem 1.1.(ii) of [1] that the number $R_n(\mathrm{Alt}(d))$ of n-dimensional irreducible projective representations of $\mathrm{Alt}(d)$ is less than $n^{2.5}$. Hence,

$$\sum_d R_n(\mathrm{Alt}(d)) < 2n^3 + 3.81n^{2.5}\log_2 n + 5n^{2.5}.$$

As above, we need to take into account the splitting of Δ-classes in the simple group by inserting a factor of n. Thus, the number of conjugacy classes of maximal subgroups of this type is bounded by $2n^4 + 3.81n^{3.5}\log_2 n + 5n^{3.5}$.

Case $\mathcal{S}_2$. We divide the sporadic groups into two sets. The first one contains those for which complete information on representation degrees is available in the Atlases. For these groups, we list the maximal multiplicity of any representation degree of the full covering group in Table 13. Adding these up will give an upper bound for the number of representations of any degree.

The other set contains the "big" sporadic groups. For these, we shall make a crude approximation based solely on conjugacy class numbers of the full covering groups. These are listed in Table 14. Adding them up

group	M_{11}	M_{12}	M_{22}	M_{23}	M_{24}	J_1	J_2	J_3	HS	McL
multiplicity	3	3	8	3	3	4	2	6	3	6

TABLE 13. Maximal multiplicities of representation degrees (over all characteristics) for the full covering groups of small sporadic groups

will give a bound for the total number of representations. The smallest representation degree of a big sporadic group is at least 12 (the group is Suz, see [2, Prop. 5.3.8]). In conclusion, the bound for the number of Δ-classes in this subfamily is 41, when $n \leq 12$, and 1988 afterwards. This needs to be multiplied by n to account for splitting.

group	J_4	He	Ru	Suz	$O'N$	Co_1	Co_2	Co_3
class number	62	33	61	210	80	167	60	42

group	Fi_{22}	Fi_{23}	Fi'_{24}	HN	Ly	Th	BM	M
class number	282	98	256	54	53	48	247	194

TABLE 14. Class numbers of the full covering groups of big sporadic groups

Case $\mathcal{S}_3$. This subfamily corresponds to cross-characteristic representations of groups of Lie type. By Corollary 1.5, the total number of equivalence classes of such representations is less than $15n$. As before, we need to add a factor of n to count for the splitting of the corresponding Δ-classes.

Case $\mathcal{S}_4$. Steinberg's tensor product theorem shows that in this case the representation is a non-trivial tensor product of twisted restricted representations. Denote $G_0 = \mathrm{P}Cl_{y^s}(q)$, a classical group of dimension y^s over the field $\mathbb{F}_q$. Now, from Corollary 6 of [33] it follows that M normalises an embedding of a classical subgroup of G_0 of the form $\mathrm{P}Cl_y(q^s)$. These embeddings are completely described in [34]. We can read from Table 1B of that work that the number of the embeddings is at most $a + 2$, where a is the number of ways to write the dimension n as a power y^s. Thus, the number of embeddings is bounded above by $\log_2 \log_2 n + 2$.

Case $\mathcal{S}_5$. In this case, the socle of M is a group $H = H_{r'}(q')$ of Lie type. The representation is characterised by a highest weight λ, as explained e.g. in [2, §5.4]. We write $V(\lambda)$ for the module of the representation corresponding to H.

Assume first that $G_0 = \mathrm{PSL}_n(q)$. Note that M does not fix a non-degenerate alternating, hermitian or quadratic form, for otherwise it would belong to Aschbacher's class $\mathcal{C}_8$.

If $H = \mathrm{PSL}_{r'+1}(q')$, we must have $q = q'$ (see [2, Prop. 5.4.6]). By Lemma 5.1, either $(r')^3/8 < n$, or the representation belongs to one of at most 3 dual pairs of representations. In the former case, r' can be one of at most $2n^{1/3}$

possibilities, and by [1, Th. 1.1.(i)], $\mathrm{PSL}_{r'+1}(q)$ has at most $n^{3.8}$ restricted representations of degree n. This gives altogether at most

$$2n^{1/3+3.8}+3$$

conjugacy classes of subgroups under Δ.

If $H = \mathrm{PSU}_{r'+1}(q')$, there is no corresponding subgroup of G. Namely, it follows from Theorems 5.4.2 and 5.4.3 presented in [2], that the dual $V(\lambda)^*$ is isomorphic to either $V(\lambda)$ or $V(\lambda)^\psi$, where ψ is the involutory automorphism of $\mathbb{F}_q = \mathbb{F}_{q'^2}$. In the first case the group would fix a non-degenerate bilinear form, and in the latter case it would fix a hermitian form (see [2, Lemma 2.10.15]). Similarly, H cannot be of type B or C as groups of these types have only self-dual representations.

If H is of one of the two remaining orthogonal types, there are at most 2 possibilites for q' ($q' = q$ or possibly $q' = q^{1/2}$). In this case, Lemma 5.1 tells us that either $(r')^8 < n$, or the representation is one of at most 2 representations. In the former case, there are at most $n^{1/3}$ possibilites for r', and by [1, Th. 1.1.(i)], the number of restricted n-dimensional representations of $H_{r'}(q')$ is at most $n^{2.5}$. Thus the number of Δ-conjugacy classes of maximal subgroups of types D and ${}^2D$, is at most $2(2n^{1/3+2.5}+2)$.

For the exceptional types, only E_6 and ${}^2E_6$ have non-self-dual representations. There are at most two possibilities for q' with the twisted type, and the number of restricted n-dimensional representations of each group is at most $n^{2.5}$ by [1, Th. 1.1.(i)]. Hence, the number of Δ-conjugacy classes is at most $3n^{2.5}$.

As a conclusion, the number of Δ-conjugacy classes has been bounded in the case $G_0 = \mathrm{PSL}_n(q)$. Multiplying by n and simplifying, we get the following bound for the number of G_0-classes:

$$b_1(n) = 2n^{5.14} + 4n^{3.84} + 3n^{3.5} + 7n.$$

Similar analysis can be done when G_0 is of any other classical type. Then, however, we have $[\Delta : G_0] \leq 8$, so we can make a simpler estimate using the fact that there are always at most two possibilities for the value of q'. The intermediate results are collected in the following table:

H	no. of Lie types	bound
$\mathrm{PSL}_{r'}(q')$	1	$4n^{1/3+3.8}+3$
$\mathrm{PSU}_{r'}(q')$	1	$4n^{1/3+3.8}+3$
other classical	4	$2n^{1/3+2.5}+2$
exceptional	9	$2n^{2.5}$

Multiplied by 8 and simplified, the bound becomes

$$b_2(n) = 64n^{4.14} + 64n^{2.84} + 144n^{2.5} + 112.$$

Notice however that, by the data given in [28], there are only at most 9 representations of simple groups of Lie type with any particular dimension n less than 32 (counting dual representations only once). Also, when $n \geq 32$, we have $b_1(n) > b_2(n)$, so we may use $b_1(n)$ as a bound for every G_0.

Conclusion. Continue assuming that the automorphism group of G is of general type (that is, $\mathrm{Aut}(G) = \mathrm{P}\Gamma(V, \mathbf{f})$). We collect into Table 15 the

partial results obtained for each Aschbacher family $\mathcal{C}_i$ and subfamily $\mathcal{S}_i$ of $\mathcal{S}$. Adding up the partial results, it is easy to check computationally that the bound given in the statement of the theorem holds for $n \geq 19$.

case	bound
subfield family $\mathcal{C}_5$	$n(\log_2 \log_2 q + 1)$
other natural families	$\frac{3}{2}n^2 + 8n \log_2 n + 8n$
$\mathcal{S}_1$	$2n^4 + 3.81n^{3.5} \log_2 n + 5n^{3.5}$
$\mathcal{S}_2$	$1988n$
$\mathcal{S}_3$	$15n^2$
$\mathcal{S}_4$	$\log_2 \log_2 n + 2$
$\mathcal{S}_5$	$2n^{5.14} + 4n^{3.84} + 6n^{3.5}$

Table 15. Upper bounds for the number of conjugacy classes of different types of maximal subgroups

For smaller values of n, we can use Lübeck's data from [28, Appendix A] to bound the number of subgroups in $\mathcal{S}_5$. We find that for all $n < 19$, this subfamily contains at most 9 representations with dimension n. Substituting this in the last row of Table 15 and adding up, we see that the given bound holds for $n \geq 8$. Finally, in his PhD thesis ([35]), Kleidman has determined all maximal subgroups of classical groups in dimension at most 11, and from this we can read that the bound given in the statement of the theorem holds even for $n < 8$. (Also, it is not difficult to check oneself that the bound holds for these groups.)

It only remains to check the groups with exceptional automorphisms. For this, we use Lemma 5.3. The number of conjugacy classes of maximal subgroups of subfield type (family $\mathcal{C}_5$) is bounded by $n(\log_2 \log_2 q + 1)$, just like in the general case. Apart from these classes, the lemma gives 5 additional classes for $G_0 = \mathrm{Sp}_4(2^k)$, and 44 classes for $G_0 = \mathrm{P}\Omega_8^+(q)$. Evidently, the statement of the theorem holds in this case as well. This concludes the proof. $\square$

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